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PASSIVATION OF UNDERACTUATED SYSTEMS WITH PHYSICAL DAMPING

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Abstract: In recent works, Interconnection and Damping Assignment Passivity-Based Control (IDA-PBC) has been succesfully applied to mechanical control problems with no physical damping present. In some cases, the friction terms can be obviated without compromising stability in closed loop. However in methods that modify the kinetic energy, a controller designed for stabilizing the undamped system might lose passivity, a key property for nonlinear system stabilization, when damping is introduced. This paper presents a necessary and sufficient condition, namely the *dissipation condition*, for recovering passivity (and hence stability) in such cases. If the *dissipation condition* is fulfilled, an IDA-PBC redesign is necessary in general, and with this goal two different methods for passivating the damped system are presented.

Keywords: Nonlinear Control, Hamiltonian Systems, Passivity Based Control

1. INTRODUCTION

Recent works in underactuated control have commonly neglected a fundamental issue: physical damping. Intuitively, the effects caused by friction in certain directions may lie outside the reach of the controller and cannot be directly cancelled (Ortega *et al.*, 2002). Possibly because of this, friction terms have been repeatedly left unmatched and the classical approach reduces to solve the control problem for an undamped open loop model, and stay in the naive hope that physical dissipation will help in some way to reach the desired equilibrium point. Nevertheless, it has been proved that in control methods that modify the kinetic energy, such as Interconnection and Damping Assignment Passivity-Based Control (IDA-PBC) (Van der Schaft, 2000) and Controlled Lagrangians (A. Bloch and Marsden, 2000), unmodeled physical damping can cause instability. In some cases, not even the tangent linearization at the equilibrium point preserves stability after the introduction of open loop damping (see (Reddy *et al.*, 2004)).

This paper shows that, under very precise conditions, friction terms can be and must be considered in the design procedure in order to shape the sum of the damping effects (natural and injected) in such a way that the whole system is dissipative. In the cases where this is possible, physical damping will actually be a guarantee for local exponential stability. Although the main problem is closed–loop stability, we will focus on the passivity property for two reasons: passivity is the cornerstone of the IDA-PBC method and further robustness analysis, and the open-loop conditions that will be presented here, are specific for passivity.

2. PROBLEM STATEMENT

The IDA-PBC method for underactuated mechanical systems aims at passivating an open–loop model of the form

$$\Sigma_1(M,V,G,R): \tag{1}$$

$$\begin{bmatrix} \dot q \\ \dot p \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & -R(q) \end{bmatrix} \begin{bmatrix} \dfrac{\partial H}{\partial q} \\ \dfrac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} u,$$

where $q \in I\!R^n$ are the generalized coordinates and $p \in I\!R^n$ the momenta, defined as $p = M\dot q$; M is the open–loop inertia matrix, $R > 0$ the physical damping matrix and the Hamiltonian is defined as

$$H = \frac{1}{2} p^\top M^{-1}(q) p + V(q).$$

Assume $G = G(q)$ has constant rank $m < n$ and hence a matrix $G^\perp$ of row rank $n - m$ exists such that

$$G^\perp G = 0, \qquad \mathrm{rank}[G^\perp | G^\top] = n.$$

In (Ortega and Spong, 2000; Ortega *et al.*, 2002; Gómez-Estern *et al.*, 2001) the IDA-PBC control problem is solved for a class of energy–preserving open–loop models (of the form $\Sigma_1(M, V, G, 0)$). In those papers, control laws are designed to transform Σ_1 into a closed–loop Hamiltonian system with dissipation of the form

$$\Sigma_2(M_d, V_d, J_d, G, R_d): \tag{2}$$
$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = (J_d(q,p) \quad - R_d(q,p)) \begin{bmatrix} \dfrac{\partial H_d}{\partial q} \\ \dfrac{\partial H_d}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} v.$$

with J_d skew-symmetric and $R_d \geq 0$. The closed–loop Hamiltonian dynamics are obtained by setting

$$J_d = \begin{bmatrix} 0 & M^{-1}M_d \\ -M_d M^{-1} & J_2(q,p) \end{bmatrix} \qquad R_d = \begin{bmatrix} 0 & 0 \\ 0 & R_2(q) \end{bmatrix}, \tag{3}$$

with $J_2 = -J_2^\top$, and then equating the open–loop and closed–loop state equations to solve the set of PDEs in the non-actuated space

$$G^\perp\{\nabla_q H + R\nabla_p H - M_d M^{-1}\nabla_q H_d + J_2 M_d^{-1} p\} = 0 \tag{4}$$

The usual approach assumes $R = 0$ (undamped open–loop model), allowing us to split (4) into p-dependent (quadratic) and p-independent terms, giving rise to the kinetic and potential energy shaping equations, namely

$$\begin{aligned} &G^\perp\left\{\nabla_q(p^\top M^{-1}p) - M_d M^{-1}\nabla_q(p^\top M_d^{-1}p)\right. \\ &\left.+2(J_2 - R_2)M_d^{-1}p\right\} = 0 && (5) \\ &G^\perp\{\nabla_q V - M_d M^{-1}\nabla_q V_d\} = 0 && (6) \end{aligned}$$

and R_2 is introduced in the subsequent damping injection step. However if $R \neq 0$ we have a third set of matching equations containing new terms that are *linear* in p, that is

$$G^\perp\{RM^{-1}p + (J_{20} - R_2)M_d^{-1}p\} = 0, \tag{7}$$

where J_{20} is a new design parameter that can be introduced by just splitting the free matrix J_2 in terms of the dependence on p as

$$J_2 = J_{20}(q) + J_{21}(q,p).$$

If this third matching equation is neglected, the closed–loop system may lose passivity and stability. Besides, the existence of a physical damping matrix R is related, as will be seen, with the following useful property:

Definition 1. (Strong dissipation). A Hamiltonian system defined on an open set $\{q \in \mathcal{X} \subset I\!R^n, p \in I\!R^n\}$ of the form Σ_2 from (2) with R_d in the form (3), is said to be *strongly dissipative* if $R_2(q) > 0\ \forall q \in \mathcal{X}$.

For such systems it is easy to check that there is a positive function $\alpha(q) > 0$ such that the rate of dissipation is

$$\dot{H}_d = -\left(\frac{\partial H_d}{\partial p}\right)^\top R_2 \frac{\partial H_d}{\partial p} < -\alpha(q)\|p\|^2$$

This is a useful property for stability analysis that in the case of underactuated systems *can only be achieved* with the aid of physical damping.

3. MAIN RESULT

In this section we will deal with four systems; Σ_1 as defined in (1), Σ_2 from (2) and the following two

$$\begin{aligned} &\Sigma_3 = \Sigma_1(M, V, G, 0) && \textit{Undamped open loop system} \\ &\Sigma_4 = \Sigma_2(M_d, V_d, J_d, G, 0) && \textit{Undamped closed loop system} \end{aligned}$$

For stabilization purposes, the IDA-PBC method calculates first an *energy shaping* law u_{es} to transform the energy–conservative system Σ_3 into Σ_4. The latter is conservative with respect to the new energy–storage function

$$H_d = \frac{1}{2}p^\top M_d^{-1}(q)p + V_d(q),$$

and has a *passive output* $y_d = G^\top \nabla_p H_d = G^\top M_d^{-1} p$.

Secondly, to turn Σ_4 into a *dissipative*, thus asymptotically stable system, a *damping injection* term $u_{di} = -K_v y_d$ must be added, leading to the form Σ_2. However, the application of the full control law $u = u_{es} + u_{di}$ to the physically damped system Σ_1 instead of Σ_3, yields

$$\dot{x} = J_d\frac{\partial H_d}{\partial x} - R\frac{\partial H}{\partial x} - \begin{bmatrix} 0 \\ G \end{bmatrix} K_v y_d.$$

where $x = [q^\top\ p^\top]^\top$. Therefore, for this system

$$\dot{H}_d = -\left(\frac{\partial H_d}{\partial x}\right)^\top R\frac{\partial H}{\partial x} - \left(\frac{\partial H_d}{\partial p}\right)^\top G K_v G^\top \frac{\partial H_d}{\partial p},$$

which does not necessarily lead to $\dot{H}_d \leq 0$ for all $K_v \geq 0$ because of the sign–indefinite first term, due to friction.

From the *passivity* point of view, a control law $u = u_{es} + v$ that passivates Σ_3 with respect to the triplet $\{H_d, v, y_d\}$ [1] may no longer yield a passive system when applied to Σ_1, i.e. when physical damping appears.

Instead of searching for the rare cases where passivity is preserved upon the addition of physical damping leaving u_{es} unchanged, we will investigate the conditions for finding a new u_{es} for which Σ_1 can be passivated for a given storage function H_d.

3.1 Passivation by interconnection assignment

Although this will be relaxed in subsequent sections, our first result applies for systems where G is constant and has the following form (possibly through variable change)

$$G = \begin{bmatrix} 0_{(n-m)\times m} \\ I_m \end{bmatrix} \tag{9}$$

In these cases we define the left anihilator of rank $n - m$ as

$$G^\perp = [I_{n-m}\ 0_{(n-m)\times m}] \tag{10}$$

Assuming the existence of a control law $u = u_{es} + v$ that transforms Σ_3 (undamped) into Σ_4, the latter being passive with respect to $\{H_d, v, y_d\}$, the following proposition establishes the condition for the existence of a state feedback $u_{es}^d \neq u_{es}$ that transforms the damped system Σ_1 into a passive system Σ_2 with respect to the same storage function H_d.

For ease of reading we define the following *linear* operator

[1] In the sequel we will denote that a system is *passive with respect to* the triplet $\{H, u, y\}$ if it is *passive* with *storage function* H, *input* v and passive *output* y.

$$J_{20} = \begin{bmatrix} \frac{1}{2}G^{\perp}(RM^{-1}M_d - M_dM^{-1}R)(G^{\perp})^{\top} & G^{\perp}RM^{-1}M_dG \\ -G^{\top}M_dM^{-1}R(G^{\perp})^{\top} & 0 \end{bmatrix} = -J_{20}^{\top} \tag{8}$$

Fig. 1. Choice of J_{20} for passivation by interconnection.

Definition 2. For $k, j \in I\!N$ with $k \leq j$, let $\psi(\cdot) : I\!R^{j\times j} \to I\!R^{k\times k}$ be the symmetric part of the k-order upper-left square submatrix of its argument, i.e. for a matrix $A \in I\!R^{j\times j}$ we have

$$\psi_k(A) = \frac{1}{2}[A + A^{\top}]_{(1\ldots k, 1\ldots k)}$$

Proposition 3. (Passivation by interconnection). Consider the system Σ_1 defined on an open set $\{q \in \mathcal{X} \subset I\!R^n, p \in I\!R^n\}$, with G in the form (9). Assume there are smooth matrices M_d, J_2 and a smooth function V_d satisfying Eqs.(5,6) with $R = R_2 = 0$, i.e. transforming Σ_3 into Σ_4.

Then, there is an energy shaping control law u_{es}^d that passivates the damped system Σ_1 in $\mathcal{X} \times I\!R^n$ with storage function H_d if and only if the following condition holds.

$$\psi_{n-m}(RM^{-1}M_d) \geq 0 \;\forall q \in \mathcal{X} \tag{11}$$

This will be called the *dissipation condition*.

Proof. (Necessity). Given a particular choice of H_d, coming as a solution of the undamped problem (for which M_d is fixed), we will investigate if there is any solution $u = u_{es} + v$ according to that makes Σ_2 passive. The parameters of such u_{es} must satisfy the extended matching conditions (5,6,7). The p-linearly dependent matching equation becomes (all functional dependences have been obviated)

$$\begin{aligned} & G^{\perp}\left[R\nabla_p H + (J_{20} - R_2)\nabla_p H_d\right] = 0 \\ & \Rightarrow G^{\perp}[RM^{-1} + (J_{20} - R_2)M_d^{-1}]p = 0 \;\; \forall p \\ & \Rightarrow G^{\perp}[RM^{-1}M_d + (J_{20} - R_2)](G^{\perp})^{\top} = 0 \\ & \Rightarrow symm\{G^{\perp}[RM^{-1}M_d - R_2](G^{\perp})^{\top}\} = 0 \\ & \Rightarrow \psi_{n-m}(R_2) = \psi_{n-m}(RM^{-1}M_d) \end{aligned}$$

because J_{20} is skew symmetric, and $G^{\perp}$ has the particular form (10) for the class of systems considered. In order to check the passivity of Σ_2 we observe that along its trajectories

$$\begin{aligned} \dot{H}_d = & -\left(\frac{\partial H_d}{\partial p}\right)^{\top} R_2 \frac{\partial H_d}{\partial p} + \left(\frac{\partial H_d}{\partial x}\right)^{\top} Gv \\ = & -p^{\top}M_d^{-1}R_2M_d^{-1}p + M_d^{-1}p\, Gv \\ = & -p^{\top}M_d^{-1}R_2M_d^{-1}p + v^{\top}y_d \end{aligned}$$

Now assume that there exists some $q^* \in \mathcal{X}$ not satisfying the dissipation condition, that is, there exists some vector $z \in I\!R^{n-m}$ such that

$$z^{\top}[\psi_{n-m}(R(q^*)M^{-1}(q^*)M_d(q^*))]z < 0$$

Defining the state $(q^*, p^*) \in (\mathcal{X} \times I\!R^n)$ with

$$p^* = M_d[z^{\top}, 0_{1\times m}]^{\top}$$

we have

$$\begin{aligned} \dot{H}_d(q^*, p^*) = & -z^{\top}[\psi_{n-m}(R_2)]z + v^{\top}y_d \\ = & -z^{\top}[\psi_{n-m}(R(q^*)M^{-1}(q^*)M_d(q^*))]z + v^{\top}y_d > v^{\top}y_d \end{aligned}$$

which means that the system is not passive in $(\mathcal{X} \times I\!R^n)$. Since this holds for any possible IDA-PBC control law u_{es}^d solution of (5,6,7), we conclude that (11) is necessary for the passivation of Σ_1 for a fixed M_d.

The proof for sufficiency is as follows. Assume that (11) holds on $\mathcal{X} \times I\!R^n$. Then we will construct an input $u = u_{es}^d + v$ that passivates system Σ_3 with storage function H_d, input v and output $y_d = G^{\top}\nabla_p H_d$. Matrix (8) cancels the non actuated terms of $RM^{-1}M_d$ outside the $(n-m)$-order upper left block and removes the skew symmetric part of the latter [2]. Hence the p-linearly dependent equation is solved with

$$R_2 = \begin{bmatrix} \psi_{n-m}(RM^{-1}M_d) & 0 \\ 0 & 0 \end{bmatrix}$$

Then, taking the matrices M_d, J_{21} and the function V_d from the solution of the undamped problem, we obtain a control law

$$\begin{aligned} u_{es}^d = & (G^{\top}G)^{-1}G^{\top}\{\nabla_q H + R\nabla_p H - M_dM^{-1}\nabla_q H_d \\ & + (J_2 - R_2)M_d^{-1}p\} + v \end{aligned}$$

such that along the closed–loop trajectories

$$\begin{aligned} \dot{H}_d = & -\left(\frac{\partial H_d}{\partial p}\right)^{\top} \psi_{n-m}(RM^{-1}M_d)\frac{\partial H_d}{\partial p} + v^{\top}y_d \leq v^{\top}y_d \\ & \forall\, (q, p) \in \mathcal{X} \times I\!R^n, \end{aligned}$$

provided that the *dissipation condition*, holds in $\mathcal{X}$. This completes the proof. ◁

This proposition provides a criterion to check if physical damping can be an obstacle to achieving passivity in closed–loop. In the positive cases it provides a simple method to construct the passivating control law. For stabilizing the passivated system it is sufficient to add a damping term of the form $v = -K_v(q, p)y_d$, with $K_v \geq 0$.

The procedure illustrated in Proposition 3 will be called *passivation by interconnection assignment*, as it exploits the interconnection matrix J_{20} to cancel the elements of $RM^{-1}M_d$ outside the critical block $\psi_{n-m}(RM^{-1}M_d)$. This procedure has the drawback of requiring exact knowledge of some elements of matrix R, which are friction parameters normally nonconstant and hard to identify experimentally.

3.2 Passivation by damping injection

An alternative approach that relaxes the parameter identification requirements is the *passivation by damping injection* method. The following proposition states the conditions for passivation in presence of physical damping via a suitable output feedback (without modifying the interconnection matrix). Again, only the dissipation condition must be satisfied by R,M and M_d for feasibility, but with strict inequality.

[2] The upper left block of J_{20} is introduced to produce a solution of (7) where R_2 is symmetric, and could be neglected without modifying the passivity result.

Let u_{es} be an energy shaping control law transforming Σ_3 into the passive system Σ_4 with storage function H_d. Then defining the passive output feedback

$$u_{di} = -\bar{R}(q,p)y_d = -\bar{R}(q,p)G^\top M_d^{-1}p,$$

with $R = R^\top \geq 0$ and applying $u = u_{es} + u_{di} + v$ to the damped system Σ_1 yields

$$\begin{aligned}
\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0 & I_n \\ -I_n & -R(q) \end{bmatrix} \begin{bmatrix} \dfrac{\partial H}{\partial q} \\ \dfrac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} u \\
&= J_d \frac{\partial H_d}{\partial x} - \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix} \frac{\partial H}{\partial x} + \begin{bmatrix} 0 \\ G \end{bmatrix} (u_{di} + v) \\
&= J_d \frac{\partial H_d}{\partial x} - \begin{bmatrix} 0 & 0 \\ 0 & RM^{-1}M_d \end{bmatrix} \begin{bmatrix} \dfrac{\partial H_d}{\partial q} \\ M_d^{-1}p \end{bmatrix} \\
&+ \begin{bmatrix} 0 \\ -G\bar{R}G^\top M_d^{-1}p \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} v \\
&= J_d \frac{\partial H_d}{\partial x} - \begin{bmatrix} 0 & 0 \\ 0 & RM^{-1}M_d - G\bar{R}G^\top \end{bmatrix} \frac{\partial H_d}{\partial x} + \begin{bmatrix} 0 \\ G \end{bmatrix} v
\end{aligned}$$

Now defining the matrices

$$C \triangleq \frac{1}{2}(RM^{-1}M_d + M_dM^{-1}R) \quad D \triangleq G\bar{R}G^\top, \tag{12}$$

we can investigate the passivity of the map $v \to y_d$ in $(\mathcal{X} \times I\!R^n)$ by computing

$$\dot{H}_d = -\left(\frac{\partial H_d}{\partial p}\right)^\top (C+D)\frac{\partial H_d}{\partial p} + v^\top y_d \tag{13}$$

that will be passive if and only if $(C + D) \geq 0$ in $\mathcal{X} \times I\!R^n$.

This new approach aims at passivating the damped system Σ_1 by simply feeding back the passive output of Σ_4, namely $y_d = G^\top M_d^{-1}p$ and leaving unchanged the energy shaping control law used to passivate the undamped model u_{es}. If the matrix $\bar{R}$ can be found, the technique is robust in the sense that any positive matrix $R' > \bar{R}$ would do the job, and hence no exact cancellation of damping terms is needed.

The following proposition provides a sufficient condition for the existence of the required $\bar{R}$. It also applies in the cases where G is q-dependent and it is not *integrable*, i.e. it cannot be transformed into a constant matrix through feedback and variable change. The idea is based in Lemma 12.31 of (Nijmeijer and Van der Schaft, 1990).

Proposition 4. (Passivation by damping injection). Assume there is an IDA-PBC control law $u = u_{es}+v$ that transforms system Σ_3 with $G = G(q)$ into a passive system Σ_4 with respect to $\{v, H_d, G^\top\nabla_p H_d\}$. Then, there exists a passivating output feedback

$$u_{di} = -\bar{R}(q)y_d$$

such that $u = u_{es} + u_{di} + v$ transforms the damped system Σ_1 into a passive system Σ_2 with $R_2 > 0$ if and only if

$$A \triangleq symm[G^\perp(RM^{-1}M_d)(G^\perp)^\top] > 0 \qquad \forall q \in \mathcal{X} \tag{14}$$

Furthermore, $\bar{R}^*$ can be taken diagonal.

Proof. (Necessity). Assume that for some q we have $A(q) \leq 0$, i.e., there is a nonzero vector x such that $x^\top Ax \leq 0$. Defining $z = (G^\perp)^\top x$ and using definitions (12) we have

$$\begin{aligned}
z^\top R_2 z &= z^\top(C+D)z = x^\top G^\perp C(G^\perp)^\top x \\
&= x^\top Ax \leq 0.
\end{aligned}$$

thus R_2 cannot be positive definite. To prove the sufficiency direction we will assume that $A(q) > 0$ $\forall q \in \mathcal{X}$. Let V be a an $m \times n$ matrix whose columns span the orthogonal complement of $C(ker\ G)$. First we prove that the $n \times n$ matrix $[V|(G^\perp)^\top]$ is nonsingular. Let $V\alpha + (G^\perp)^\top\beta = 0$, with $\alpha \in I\!R^m$ and $\beta \in I\!R^{n-m}$. Then

$$0 = G^\perp C(V\alpha + (G^\perp)^\top\beta) = G^\perp C(G^\perp)^\top\beta = A\beta$$

Since A is assumed to be positive definite, this implies that $\beta = 0$ and $\alpha = 0$. Then we observe that

$$\begin{aligned}
&[V|(G^\perp)^\top]^\top(C+D)[V|(G^\perp)^\top] = \\
&= \begin{bmatrix} V^\top CV + V^\top DV & 0 \\ 0 & G^\perp C(G^\perp)^\top \end{bmatrix}
\end{aligned}$$

Since rank $V^\top DV =$

$$\text{rank } [V|(G^\perp)^\top]^\top D[V|(G^\perp)^\top] = \text{rank } D$$

we conclude that as $D = G\bar{R}G^\top$, $R_2 = C + D$ can be made positive definite by choosing an appropriate $\tilde{R} = \tilde{R}^\top$ (if necessary diagonal). This would give a *strongly dissipative* closed–loop system. ◁

3.3 Local exponential stability

From Proposition 4 it is clear that if the dissipation condition holds strictly, the system can be made strongly dissipative by damping injection. But for this condition to hold, it is necessary that $\det(R) \neq 0$. Hence strong dissipation is a property exclusive to physically damped systems. This feature is very convenient for stability analysis, because as will be shown, strongly dissipative systems are locally exponentially stable (LES). To illustrate this point we will analyze the Jacobian linearization close to the origin, namely

$$\begin{bmatrix} \dot{z}_q \\ \dot{z}_p \end{bmatrix} = \begin{bmatrix} 0 & M^{-1} \\ -M_dM^{-1}\dfrac{\partial^2 V_d}{\partial q^2} & (J_{20} - R_2)M_d^{-1} \end{bmatrix}_{x=0} \begin{bmatrix} z_q \\ z_p \end{bmatrix}$$

Asymptotic stability of this system will be investigated by defining the positive Lyapunov function

$$V = \frac{1}{2}z^TQz, \quad Q \triangleq \frac{\partial^2 H_d}{\partial z_p^2}(0) = \begin{bmatrix} \dfrac{\partial^2 V_d}{\partial q^2}(0) & 0 \\ 0 & M_d^{-1}(0) \end{bmatrix}$$

Clearly, $Q > 0$ in a well designed controller. The time derivative of V is

$$\begin{aligned}
\dot{V} &= z^\top Q\dot{z} \\
&= z_p^\top M_d^{-1}(0)[J_2(0) - R_2(0)]M_d^{-1}(0)z_p \\
&= -z_p^\top M_d^{-1}(0)R_2(0)M_d^{-1}(0)z_p < 0\ , \qquad \forall z_p \neq 0
\end{aligned}$$

As we have built a positive definite R_2, the linearized system will converge asymptotically to the largest invariant set where $z_p \equiv 0$. This set is such that

$$\dot{z}_p = 0 \Rightarrow -M_dM^{-1}\frac{\partial^2 V_d}{\partial q^2}(0)z_q = 0 \Rightarrow z_q = 0$$

hence linear asymptotic stability is a fact and local exponential stability is the corollary.

4. EXAMPLE: BALL ON BEAM

This system has been successfully addressed in the IDA-PBC framework (see (Ortega *et al.*, 2002)). Yet, physical dissipation has been neglected. Besides the risk of instability, the closed–loop dissipation matrix is not full rank, a situation leading to cumbersome stability proofs and not ensuring local exponential stability.

4.1 System model

The commonly used physical model, under some time and constant scaling (Ortega *et al.*, 2002), the Euler Lagrange equations become

$$\begin{aligned} \ddot{q}_1 + g\sin(q_2) \quad q_1\dot{q}_2^2 + \beta_1(q,p)\dot{q}_1 &= 0 \qquad (15)\\ (L^2+q_1^2)\ddot{q}_2 + 2q_1\dot{q}_1\dot{q}_2 + gq_1\cos(q_2) + \beta_2(q,p)\dot{q}_2 &= u, \end{aligned}$$

where q_1 is the position of the ball on the beam and q_2 is the angle of the bar, with the origin at the horizontal position. Here we have introduced the positive damping functions β_1 and β_2 as suggested in (Reddy *et al.*, 2004) but here we also admit the possibility of some dependence on the state.

4.2 Stability of the standard IDA-PBC controller

In (Ortega *et al.*, 2002) an IDA-PBC control law was developed for a damping-free model (i.e setting $\beta_i(q,p) = 0$ in (15)), which is not included here for the sake of brevity. As expected from the preceding arguments, the closed–loop dissipation matrix is not full rank,

$$R_2 = \begin{bmatrix} 0 & 0 \\ 0 & k_v \end{bmatrix}$$

and the asymptotic stability analysis is nontrivial, as when the derivative of the closed–loop Hamiltonian

$$\dot{H}_d = \quad k_v \left(\frac{p_1\sqrt{L^2+q_1{}^2} \quad p_2\sqrt{2}}{(L^2+q_1{}^2)^{3/2}} \right)^2,$$

reaches zero, p need not be zero.

4.3 Physical damping and nonlinear damping injection

For any positive values of $\beta_i(q,p)$, the *dissipation condition* is trivially satisfied *globally*. Hence it is always possible to inject enough damping to overcome this difficulty. Here we will design the damping injection terms to get a globally positive definite closed–loop dissipation matrix. The novelty with respect to (Ortega *et al.*, 2002), where linear damping injection was used, appears in the nonlinear output feedback

$$u_{di} = \quad k_v(q,p)y_d = \quad k_v(q,p)G^\top \nabla_p H_d$$

that is needed for the system to be globally *strongly dissipative*. Actually this happens when the closed–loop dissipation matrix is globally positive, that is,

$$k_v > \frac{1}{2\sqrt{2}} \frac{6\,\beta_1 \quad L^2+q_1{}^2)\,\beta_2 + \beta_1{}^2 \quad L^2+q_1{}^2)^2 + \beta_2{}^2}{\sqrt{L^2+q_1{}^2}}$$

This can be satisfied with a constant k_v^* on any compact set. However, for $q \in I\!R^n$ there is no constant output feedback satisfying this equation, thus we recourse to a state dependent form of k_v like

$$u_{di} = \quad \frac{\bar{\beta}_1^2 \quad L^2+q_1{}^2)^2 + \bar{\beta}_2^2}{2\sqrt{L^2+q_1{}^2}} \left(\frac{p_1\sqrt{L^2+q_1{}^2} \quad p_2\sqrt{2}}{(L^2+q_1{}^2)^{3/2}} \right) \qquad (16)$$

where

$$\bar{\beta}_1 > \max_{(q,p)}(\beta_1(q,p)) \qquad \bar{\beta}_2 > \max_{(q,p)}(\beta_2(q,p))$$

are some *estimated* upper bounds on the friction parameters. With this controller parameters, the closed–loop system is locally exponentially stable and in virtue of the strong dissipation property it can be easily proved that the trajectories converge to the set

$$\{q \in I\!R^n | \nabla V_d(q) = 0\} \cap \{p = 0\}$$

which is a countable set of isolated points of the form

$$\bar{q} = (L\sinh(\sqrt{2}i\pi), i\pi)\ ,\ i \in I\!N$$

including the origin and other points outside $\{q_2 \in (\pi, \pi)\}$. This results significantly simplifies the stability proofs.

5. SIMULATIONS

System (15) has been simulated with the energy shaping control law from (Ortega *et al.*, 2002) and the two possible damping injection terms discussed in Section 4.3. First, we will use a constant linear feedback as proposed in (Ortega *et al.*, 2002) (setting $k_v > 0$ constant) and secondly the nonlinear output feedback (16). While for sufficiently large k_v both controllers will work fine locally, for initial conditions further away from the origin the linear output feedback will be insufficient to keep H_d always decreasing, whereas (16) ensures global dissipation.

Figure 2 depicts the simulation results of the ball and beam under different dissipation conditions. The three graphs in the upper row show the trajectories of q_1 and q_2 vs. time. The lower row shows the time dependence of the closed–loop Hamiltonian function H_d corresponding to each trajectory in the above graph. Three different conditions have been simulated. The first case, (graphs (a1) and (a2)), illustrates the IDA-PBC controller with k_v constant acting on a damping-free model, as in (Ortega *et al.*, 2002). For any $k_v > 0$, the semidefinite dissipation matrix is sufficient to ensure stability and no further considerations must be done. The second simulation, (b1) and (b2) shows how the performance of the constant k_v controller is downgraded when physical damping is introduced in the model and not considered for design. Figure (b2) has been zoomed in to stress out that the closed–loop energy is not monotonic: stability can be compromised. Graphs (c1) and (c2) show the closed–loop behavior of the physically damped system when the nonlinear damping term (16) is added to the controller. This controller recovers a monotonic Lyapunov function for every initial state, even without exact knowledge of the β parameters.

6. A NEGATIVE EXAMPLE

In order to highlight how critical the *dissipation condition* can be, we have searched for a negative example for which (14) is never fulfilled for any choice of M_d compatible with the requirements of the IDA-PBC method. Such a pathology is found in the Inertia Wheel Pendulum, for which the IDA-PBC control problem has been easily solved in the undamped case. While not going into detail for brevity, we refer the reader to (Acosta *et al.*, 2004), where the conditions for IDA-PBC stability can be summarized as

(i) $m_{11}^0 > 0,\ m_{11}^0 m_{22}^0 > \quad m_{12}^0)^2$.
(ii) $m_{11}^0 + m_{12}^0 < 0$.

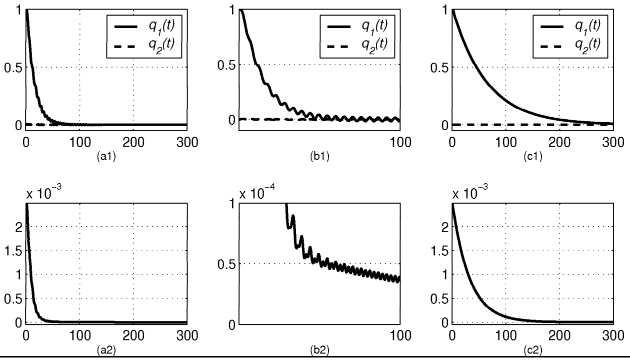

Fig. 2. Simulation results for the Ball and Beam. Upper row: Position vs. time. Lower row: Energy vs. time.

where m_{ij}^0 are the elements of the 2×2 closed–loop inertia matrix M_d obtained in the kinetic energy shaping step. Condition (i) yields a positive inertia matrix, while (ii) is related to the minimum assignment of the closed–loop potential energy. If we introduce physical damping in the problem, the following proposition stems.

Proposition 5. For the Inertia Wheel Pendulum, with dynamics described in (Acosta *et al.*, 2004), there is no matrix M_d achievable by the IDA-PBC method as proposed in (Ortega and Spong, 2000), satisfying (14).

Proof. Condition (14), for passivating the Inertia Wheel Pendulum yields

$$G^{\perp}RM\ {}^{1}M_d(G^{\perp})^{\top} > 0 \Rightarrow$$

$$\eta^2 \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} r_1 + r_2 & r_2 \\ r_2 & r_2 \end{bmatrix} \begin{bmatrix} m_{11}^0 & m_{12}^0 \\ m_{12}^0 & m_{22}^0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} > 0$$

$$\Rightarrow \eta^2 r_1 (m_{11}^0 + m_{12}^0) > 0,$$

and this inequality contradicts the stability condition (ii). ◁

As a consequence, standard IDA-PBC for this system with physical damping is not possible.

Remark 6. In (Acosta *et al.*, 2004), M_d was given explicitly for with a free parameter $\Psi(q_1)$. In that paper the undamped Inertia Wheel problem was solved simply setting $\Psi(q_1)$ to zero—yielding a constant M_d. Here, we prove that even with a nonzero free parameter $\Psi(q_1) \neq 0$, the Inertia Wheel pendulum with natural damping cannot be passivated.

Proof. Choosing the most general form for the closed–loop inertia matrix given in (Acosta *et al.*, 2004), we have

$$M_d(q_1) = \int_{q_{1*}}^{q_1} \Psi(\mu) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} d\mu + \begin{bmatrix} m_{11}^0 & m_{12}^0 \\ m_{12}^0 & m_{22}^0 \end{bmatrix}.$$

Checking Assumption A.1 from (Acosta *et al.*, 2004), regarding the existence of solution for the kinetic energy PDE, it is easy to see that the $\Psi(q_1)$—dependent term of $M_d(q_1)$ cancels. Thus the constant η is the same as the one given in the proof of Proposition 5 and therefore, the stability Assumption A.2 from (Acosta *et al.*, 2004) remains unchanged, i.e. the condition (ii) given in Proposition 5. ◁

7. CONCLUSIONS

In this paper the IDA-PBC control technique for underactuated mechanical systems has been revised to incorporate an important phenomenon that has been neglected in previous related works: open–loop damping. Given a solution of the IDA-PBC matching equations for the undamped model, this paper provides necessary and sufficiency conditions for passivating the damped system. An interesting open issue is the application of the previous results to nonsmooth friction forces, like Coulomb friction. In this case the open–loop damping matrix R tends to infinity at the zero crossings of p and the system cannot be turned dissipative by feedback with the illustrated method. A novel approach specific for this case is under study.

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